# Descriptive Set Theory
## Lecture 18

**Corollary.** If $X$ is Polish, then $|\mathcal{B}(X)| \le 2^{\aleph_0}$.

**Proof.** $|\mathcal{B}(X)| = |\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha(X)|$. $|\Sigma^0_1(X)| \le 2^{\aleph_0} \Rightarrow |\Pi^0_1(X)| \le 2^{\aleph_0}$.
Also, $\forall \alpha < \omega_1$, $|\Sigma^0_\alpha(X)| = \boxed{\le 2^{\aleph_0} \mid \le 2^{\aleph_0} \mid \cdots}\ |(2^{\aleph_0})^{\alpha}| \le 2^{\aleph_0 \times \aleph_0}$
$= 2^{\aleph_0}$, so by induction, each $\alpha$ $|\Sigma_\alpha(X)| \le 2^{\aleph_0}$.
Finally, $|\mathcal{B}(X)| = |\omega_1 \times \Sigma^0_\alpha(X)| \le \omega_1 \times 2^{\aleph_0} \le |2^{\aleph_0} \times 2^{\aleph_0}|$
$= 2^{\aleph_0}$. $\square$

**Closure properties.** Let $X$ be a top. space and $1 \le \alpha < \omega_1$.

(a) $\Sigma^0_\alpha(X)$ is closed under ctbl unions.
$\Pi^0_\alpha(X)$ is — — — — — — ctbl intersections.
$\Delta^0_\alpha(X)$ is — — — — — — complements.

(b) If $X$ is metrizable, then
$\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$ are closed under finite unions and intersections.

**Proof.** (a) is by the def. For (b), we show that if $A, B \in \Sigma^0_\alpha(X)$ then so is $A \cap B$. Let $A = \bigcup_n A_n$, $B = \bigcup_m B_m$, where $A_n \in \Pi^0_{\alpha_n}(X)$ and $B_m \in \Pi^0_{\beta_m}(X)$, $\alpha_n, \beta_m < \alpha$. Then $A \cap B = \bigcup_{n,m} A_n \cap B_m$. Then $A_n \cap B_m$

$\in \Pi^0_{\max(\alpha_n,\beta_m)}(X)$ (this is where metrizability is used)

so $A \cap B$ is still in $\Sigma^0_\alpha(X)$. $\square$

**Example.** Recall that $C^1[0,1]$ is the subset of $C[0,1]$ of all continuously differentiable functions.
We show that $C^1[0,1]$ is $\Pi^0_3(C[0,1])$.
It is not hard to check using uniform continuity of derivatives that for $f \in C[0,1]$, $f \in C^1[0,1] \iff$

$$\underbrace{\forall \varepsilon \in \mathbb{Q}^+ \underbrace{\exists \underbrace{(I_0,\dots,I_n)}_{\substack{\text{a tuple of}\\ \text{rational intervals}\\ \text{covering } [0,1]}} \underbrace{\forall k \le n \ \forall a \ne b, c \ne d \in I_k \underbrace{\left|\frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(d)}{c-d}\right| < \varepsilon}_{\text{closed, i.e. } \Pi^0_1}}_{\text{still closed, i.e. } \Pi^0_1}}_{\Sigma^0_2}}_{\Pi^0_3}$$

Let $\Sigma^0_\alpha$, $\Pi^0_\alpha$, $\Delta^0_\alpha$ be the classes of all subsets of Polish spaces that are from the corresponding level of the Borel hierarchy.

**More closure properties.** Let $\Gamma$ be one of $\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha$ classes for $1 \le \alpha < \omega_1$.

(a) $\Gamma$ is closed under continuous preimages, i.e. if $X, Y$ are Polish spaces and $f: X \to Y$ is continuous, then $f^{-1}(\Gamma(Y)) \subseteq \Gamma(X)$.

(b) $\Gamma$ is closed under putting sets on ctbly many different fibers, i.e. if $A_n \in \Gamma(X)$, $X$ Polish, then the set $A = \bigcup_{n \in \mathbb{N}} A_n \times \{n\} \in \Gamma(X \times \mathbb{N})$.

(c) If $A \in \Gamma(X \times Y)$ then $A_{x_0}$ and $A^{y_0}$ are still in $\Gamma$, for any fixed $x_0 \in X$ and $y_0 \in Y$.

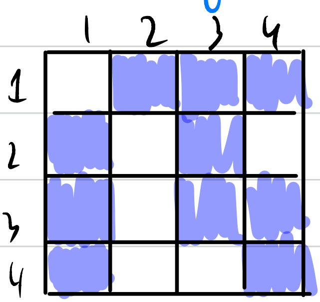
**Proof.** (a) Induction on $\alpha$ using that $f^{-1}$ commutes with complement and unions.

(b) Also induction on $\alpha$ using that fact that union and complement commute with taking a fiber, and the fact that the statement is true for open sets.

(c) This follows from the fact that the maps $y \mapsto (x_0, y)$ and $x \mapsto (x, y_0)$ are continuous and $\Gamma$ class is closed under continuous preimages.

□

Next we prove that the hierarchy is strict, i.e. $\Delta^0_\alpha \subsetneq \Sigma^0_\alpha$. We will use universal sets for these classes and Cantor diagonalization.



**Cantor diagonalization** Let $X$ be a set and $R \subseteq X \times X$. Then $\mathrm{AntiDiag}(R) := \{x \in X : (x,x) \notin R\}$ is not equal to $R_{x_0}$ or $R^{x_0}$ for any $x_0 \in X$.

**Proof.** For any $x_0 \in X$,

$$x_0 \in \mathrm{AntiDiag}(R) \iff (x_0, x_0) \notin R \iff x_0 \notin R_{x_0} \text{ and } x_0 \notin R^{x_0}.$$

□

**Def.** For Polish spaces $X, Y$, we say that a set $U \subseteq X \times Y$ *parametrizes* $\Gamma(Y)$ iff $\{U_x : x \in X\} = \Gamma(Y)$. We say that $U$ is an *$X$-universal set* for $\Gamma(Y)$ if it parametrizes $\Gamma(Y)$ and is itself in $\Gamma(X \times Y)$.

**Prop.** For all ctbl ordinals $\alpha \geq 1$, the classes $\Sigma^0_\alpha$ and $\Pi^0_\alpha$ admit $2^{\mathbb{N}}$-universal sets.

**Proof.** Because complements commute with fibers, complements of $\Sigma^0_\alpha$ universal sets are $\Pi^0_\alpha$ universal and vice versa. We build a universal set for $\Sigma^0_\alpha$, by induction.
For $\alpha = 1$, we need to build an open set $\mathcal{U} \subseteq 2^{\mathbb{N}} \times Y$ s.t. $\{\mathcal{U}_x\}_{x \in 2^{\mathbb{N}}}$ is all open subsets of $Y$.
Fix a ctbl basis $(V_n)$ for $Y$, so each open set $V \subseteq Y$ has its code $c(V) := \{n \in \mathbb{N} : V_n \subseteq U\}$.
Define, for each $x \in 2^{\mathbb{N}}$,

$$\mathcal{U}_x := \bigcup_{\substack{n \in \mathbb{N} \\ x(n) = 1}} V_n.$$

Thus it's clear that $\mathcal{U} \subseteq 2^{\mathbb{N}} \times Y$ parametrizes $\Sigma^0_1(Y)$.
$\mathcal{U}$ is open because $\forall (x,y) \in \mathcal{U}$ then $y \in V_n$ for some $n \in \mathbb{N}$ s.t. $x(n) = 1$. But then

$$[\underbrace{* * \ldots * 1}_{n}] \times V_n \subseteq \mathcal{U}.$$

Now suppose that universal sets exist for all $\Sigma^0_\beta(Y)$ (hence also $\Pi^0_\beta(Y)$) classes for all $\beta < \alpha$.
We define one for $\Sigma^0_\alpha(Y)$. Let $\alpha_n < \alpha$ be s.t.

$\sup_n \{\alpha_n+1\} = \alpha$. Then any set $A \subseteq \Sigma^0_\alpha(Y)$ is a union $\bigcup_n B_n$ where each $B_n \in \Pi^0_{\alpha_n}(Y)$.
Let $U^{(n)} \subseteq 2^{\mathbb{N}} \times Y$ be a universal set for $\Pi^0_{\alpha_n}(Y)$.
It is enough to build a $2^{\mathbb{N}\times\mathbb{N}}$-universal set for $\Sigma^0_\alpha(Y)$ because $2^{\mathbb{N}\times\mathbb{N}}$ is homeomorphic to $2^{\mathbb{N}}$.
Then define, for $x \in 2^{\mathbb{N}\times\mathbb{N}}$

$$U_x := \bigcup_{n\in\mathbb{N}} U^{(n)}_{x(n)},$$ where $x(n)$ is the $n^{th}$ row of $x$.

Again it's clear that $U$ parametrizes $\Sigma^0_\alpha(Y)$.
To see that $U \in \Sigma^0_\alpha(2^{\mathbb{N}\times\mathbb{N}} \times Y)$, note that

$$(x,y) \in U \iff y \in \bigcup_{n\in\mathbb{N}} U^{(n)}_{x(n)} \iff \exists n \in \mathbb{N}\ (x(n),y) \in U^{(n)}.$$

But the set $\{(x,y) \in 2^{\mathbb{N}\times\mathbb{N}} \times Y : (x(n),y) \in U^{(n)}\}$ is the preimage of $U^{(n)}$ under $(x,y) \mapsto (x(n),y)$ which is a projection, hence continuous, hence this set too is in $\Pi^0_{\alpha_n}(2^{\mathbb{N}\times\mathbb{N}} \times Y)$. Thus, $U$ is a union of sets from $\Pi^0_{\alpha_n}$, $n\in\mathbb{N}$, classes, hence in $\Sigma^0_\alpha(2^{\mathbb{N}\times\mathbb{N}} \times Y)$. $\square$